Traffic Assignment II:
Equilibrium and day-to-day dynamics

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Modeling and Simulation of Transportation Networks

July 29, 2015

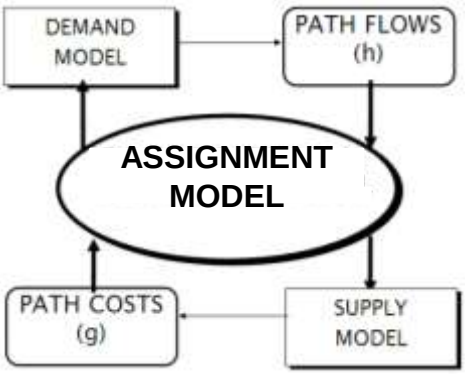
Department of Transportation Engineering
University of Naples "Federico II"

OUTLINE

- INTRODUCTION
- SUPPLY MODELS
- DEMAND MODELS
- **SUPPLY/DEMAND INTERACTION MODELS (Assignment models)**

ASSIGNMENT MODELS

SIMULATE DEMAND-SUPPLY INTERACTIONS, RESULTING FLOWS AND PERFORMANCES ON NETWORK ELEMENTS.



ASSIGNMENT MODELS
Classification factors

- **SUPPLY MODELS**
  - ✓ WITHIN-DAY STATIC
  - ✓ WITHIN-DAY DYNAMIC
  - ✓ …
- **DEMAND MODELS**
  - ✓ WITHIN-DAY STATIC
  - ✓ WITHIN-DAY DYNAMIC
  - ✓ …
- **ASSIGNMENT MODELS**
  - ✓ Not Congested Systems
  - ✓ Congested Systems
    - ➢ Equilibrium
    - ➢ Dynamic Processes

ASSIGNMENT MODELS
Classification factors

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ASSIGNMENT MODELS
Congestion

Occurs in most transportation systems, generally when multiple users interact with each other

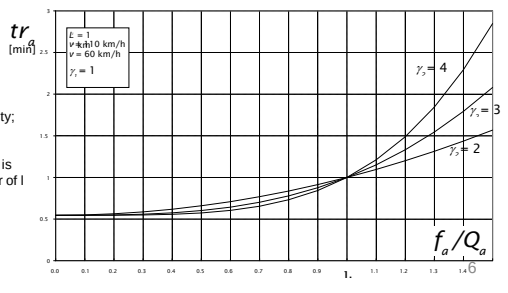
Worsening the overall performances, such as the mean speed or the travel time, since a vehicle may not be able to move at the desired speed

Example: motorway links *(see Lecture: Traffic Performance IV)*

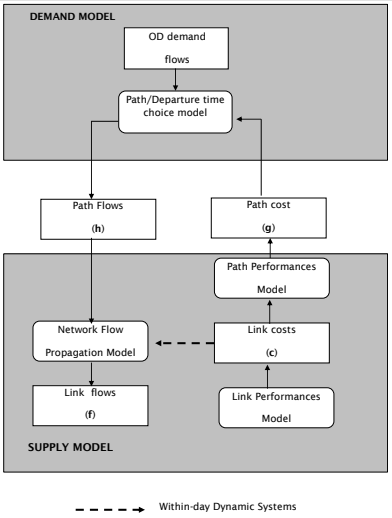
$$tr_a(f_a) = \frac{L_a}{v_{0a}} + \gamma_1 \left(\frac{L_a}{v_{ca}} - \frac{L_a}{v_{0a}} \right) \left(\frac{f_a}{Q_a} \right)^{\gamma_2}$$

where:

L_l	is the length of link l;
v_{0l}	is the free-flow average speed;
v_{cl}	is the average speed with flow equal to capacity;
Q_l	is the link capacity, i.e. the average maximum number of equivalent vehicles that can travel along the road section in a time unit. Capacity is usually obtained as the product of the number of lanes on the link l, N_l, and lane capacity, Q_{ul};
γ_1, γ_2	*are parameters of the function.*



ASSIGNMENT MODELS
Not congested network



taken from:
Cascetta (2009). *Transportation System Analysis: models and applications.* 2nd edition. Springer.

ASSIGNMENT MODELS
Not congested network

	WITHIN-DAY STATIC	WITHIN-DAY DYNAMIC
Supply	$\mathbf{c} = \mathbf{c}^{\circ}$	$t_j = t_j^0$ $TT_j = \sum_{m \geq j} \Delta^T(j,m)\ t_m^0$
	$\mathbf{g} = \Delta^T \mathbf{c}$	$g_j = TT_j + ELAD_j(TT_j)$
	$\mathbf{f} = \Delta\ \mathbf{h}$	$f_j = \sum_{l \leq j} \Delta(l,j) * h(l)$ $\Delta_{l,j} = \Gamma(t_l^0, \ldots, t_j^0)$
Demand	$\mathbf{h} = \boldsymbol{P}(\mathbf{g})\ d$	$h_j = P_j(\mathbf{g})\ d$
	$\mathbf{f} = \Delta\ \boldsymbol{P}(\Delta^T \mathbf{c}^{\circ})\ d$	$f_j = \sum_{l \leq j} \Delta_{l,j}\ P_j(\mathbf{g})\ d$ $g_j = \Gamma(\mathbf{t}^{\circ}) + ELAD_j(\Gamma(\mathbf{t}^{\circ}))$

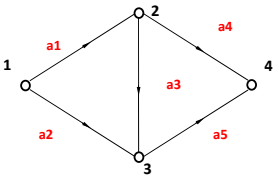
ASSIGNMENT MODELS
Acronyms for uncongested traffic assignment models in the literature

DNL	*Deterministic Network Loading or "All or Nothing" (AoN)*
SNL	*Stochastic Network Loading*
DDNL	*Dynamic Deterministic Network Loading*
DSNL	*Dynamic Stochastic Network Loading*

ASSIGNMENT MODELS
Not congested network

Example

Let us consider OD pair 1-4 in the following network



Demand

d(1,4)=150 veic/h

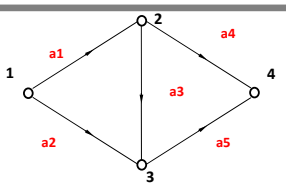
Incidence matrix

O-D paths / links	1-4: 1	2	3
a_1	1	0	1
a_2	0	1	0
a_3	1	0	0
a_4	0	0	1
a_5	1	1	0

ASSIGNMENT MODELS
Not congested network

Example

Free flow cost



Path choice model $P(g_k) = \frac{\exp[-(1/60)\cdot g_k]}{\sum_{k'} \exp[-(1/60)\cdot g_{k'}]}$

Link cost function $c(f) = c_0 + (f/Cap)^2$ $Cap = 200$ veic/h

Demand d(1,4)=150 veic/h

	c_0	Path #	$g = D^T c$	$P(g)$	$h = Pd$		$f = Dh$	$c = c(f)$
a1	5,0	**1**	15	33%	50	**a1**	100	5,3
a2	10,0	**2**	15	33%	50	**a2**	50	10,1
a3	5,0	**3**	15	33%	50	**a3**	50	5,1
a4	10,0					**a4**	50	10,1
a5	5,0					**a5**	100	5,3

$c(f) \sim c_0$ ➔ not congested network

ASSIGNMENT MODELS
Classification factors

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ASSIGNMENT MODELS
Congested network

Example

Demand d(1,4)= 1500 veic/h

	c_0	Path #	$g= D^T c$	$P(g)$	$h=Pd$		$f=Dh$	$c=c(f)$
a1	5,0	**1**	15	33%	500	**a1**	1000	30,0
a2	10,0	**2**	15	33%	500	**a2**	500	16,3
a3	5,0	**3**	15	33%	500	**a3**	500	11,3
a4	10,0					**a4**	500	16,3
a5	5,0					**a5**	1000	30,0

	c	Path #	$g=D^T c$	$P(g)$	$h=Pd$		$f=Dh$
a1	30,0	**1**	71	25%	372	**a1**	936
a2	16,3	**2**	46	38%	564	**a2**	564
a3	11,3	**3**	46	38%	564	**a3**	372
a4	16,3					**a4**	564
a5	30,0					**a5**	936

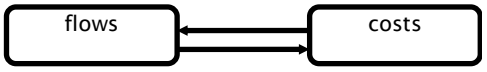
$c(f) \neq c_0$

$f^{k-1} \neq f^k$ ⇒ **??**

ASSIGNMENT MODELS
Congested network

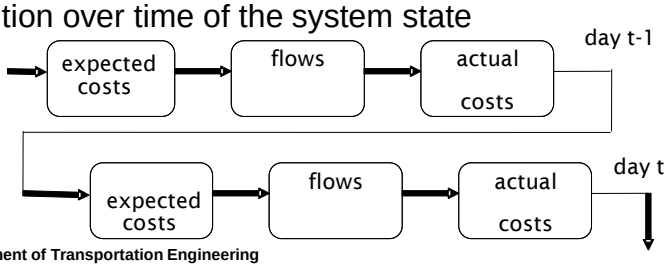
DAY TO DAY STATIC (equilibrium)

Mutually consistent flows and costs



DAY TO DAY DYNAMIC (dynamic processes)

Evolution over time of the system state



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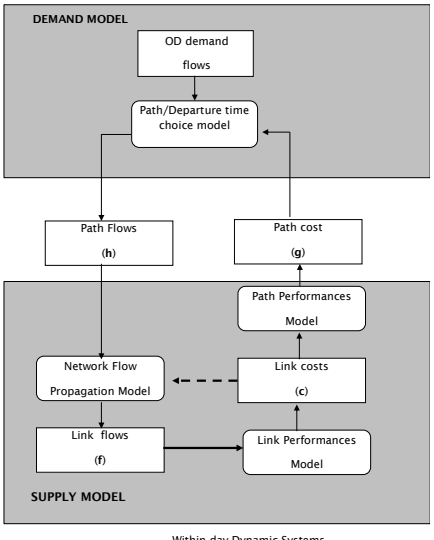
ASSIGNMENT MODELS
Congested network – Equilibrium models



ASSIGNMENT MODELS
Congested network – Equilibrium models

	WITHIN-DAY STATIC	WITHIN-DAY DYNAMIC
Supply	$\mathbf{c} = \mathbf{c}(\mathbf{f})$	$\mathbf{t} = \mathbf{t}(\mathbf{f})$ $TT_j = \sum_l \Delta^T(l,j)\mathbf{t}(f_l)$
	$g_i = \Delta_i^T \mathbf{c}$	$g_j = TT_j + ELAD_j(TT_j)$
	$\mathbf{f} = \Delta\, \mathbf{h}$	$f_j = \sum_{l \le j} \Delta(l,j) * \mathbf{h}(l)$
		$\Delta_{l,j} = \Gamma(t_l, \ldots, t_j)$
Demand	$\mathbf{h} = P(\mathbf{g})\ d$	$h_j = P(\mathbf{g})\ d$
	$\mathbf{f} = \Delta\, P(\Delta^T \mathbf{c})\ d$ $\mathbf{c} = \mathbf{c}(\mathbf{f})$	$f_j = \sum_{l \le j} \Delta_{l,j}(t_l, \ldots, t_j) P_j(\mathbf{g})\ d$ $g_j = TT_j + ELAD_j(TT_j)$ $\mathbf{t} = \mathbf{t}(\mathbf{f})$
	$\mathbf{f}^* = \Delta\, P(\Delta^T \mathbf{c}(\mathbf{f}^*))\ d$	

FIXED-POINT MODELS

$$f_j^* = \sum_{l \le j} \Delta_{l,j}\ [t(f^*)]\ Pj\{\sum_{l \le j} [\Delta_{l,j}\ [t(f^*)]\ t(f_l^*) + ELAD(\Gamma(t(f^*_l)))]\}d$$

ASSIGNMENT MODELS
Acronyms for equilibrium assignment models in the literature

DUE *Deterministic User Equilibrium*

SUE *Stochastic User Equilibrium*

DDUE *Dynamic Deterministic User Equilibrium*

DSUE *Dynamic Stochastic User Equilibrium*

ASSIGNMENT MODELS
Congested network – Equilibrium models

MSA-FA ALGORITHM

Algorithm structure:

- k=0 (inizialization)
- $c^0=c(f=0)$
- $f^0=f^0_{SNL}$

- $k=k+1$
- $c^k=c(f^{k-1})$
- $f^k = (k-1)/k\ f^{k-1} + (1/k)\ f^k_{SNL}$
- convergence criterion check

$f^k_{SNL}=f^{k-1}$

Note: faster convergence reached if k reset after some iterations
Optimal $k\in$[5-10]

ASSIGNMENT MODELS
Congested network – Equilibrium models

MSA-FA ALGORITHM

Example (1/4)

k=0 (init)	c0	Path #	g	P	h		f_snl	f^0
a1	5.0	1	15	33%	2667	a1	5333	5333
a2	10.0	2	15	33%	2667	a2	2667	2667
a3	5.0	3	15	33%	2667	a3	2667	2667
a4	10.0					a4	2667	2667
a5	5.0					a5	5333	5333

k=1	$c(f^{k-1})$	Path #	g	P	h		f_snl	f^1
a1	716.1	1	1615	0%	0	a1	4000	4000
a2	187.8	2	904	50%	4000	a2	4000	4000
a3	182.8	3	904	50%	4000	a3	0	0
a4	187.8					a4	4000	4000
a5	716.1					a5	4000	4000

taken from:
Cascetta (2009). *Transportation System Analysis: models and applications.* 2nd edition. Springer.

ASSIGNMENT MODELS
Congested network – Equilibrium models

MSA-FA ALGORITHM

$f^2 = (1/2)\, f^1 + (1/2)\, f^2_{SNL}$

Example (2/4)

k=2	$c(f^{k-1})$	Path #	g	P	h		f_snl	f^2
a1	405.0	1	815	32%	2520	a1	5260	4630
a2	405.0	2	810	34%	2740	a2	2740	3370
a3	5.0	3	810	34%	2740	a3	2520	1260
a4	405.0					a4	2740	3370
a5	405.0					a5	5260	4630

$f^3 = (2/3)\, f^2 + (1/3)\, f^3_{SNL}$

k=3	$c(f^{k-1})$	Path #	g	P	h		f_snl	f^3
a1	541.0	1	1127	0%	28	a1	4014	4425
a2	288.9	2	830	50%	3986	a2	3986	3575
a3	44.7	3	830	50%	3986	a3	28	850
a4	288.9					a4	3986	3575
a5	541.0					a5	4014	4425

ASSIGNMENT MODELS
Congested network – Equilibrium models

MSA-FA ALGORITHM

Example (3/4)

…

iteration k

k	$c(f^{k-1})$		g	P	h		f_snl	f^k
a1	459.5	h1	931.0	7%	527	a1	4264	4264
a2	354.0	h2	813.5	47%	3736	a2	3736	3736
a3	12.0	h3	813.5	47%	3736	a3	527	528
a4	354.0					a4	3736	3736
a5	459.5					a5	4264	4264

$c(f^{k-1}) = c(f^k)$

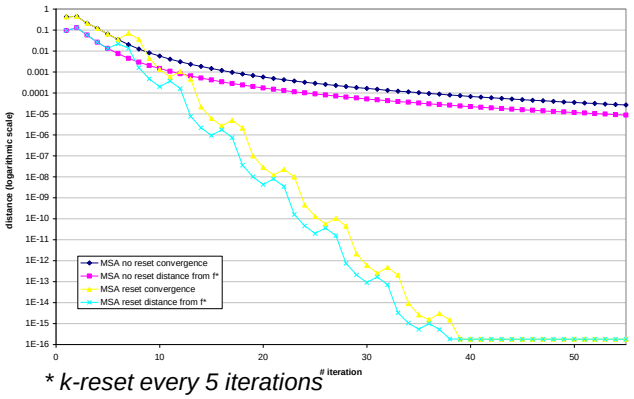
$f^{k-1} = f^k$ ⇨ **equilibrium**

ASSIGNMENT MODELS
Congested network – Equilibrium models

MSA-FA ALGORITHM

Example (4/4)

Distance from equilibrium vector **f*** vs. convergence test $|\mathbf{f}^{k}_{SNL}-\mathbf{f}^{k-1}|/|\mathbf{f}^{k-1}|$



** k-reset every 5 iterations*

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Classification factors

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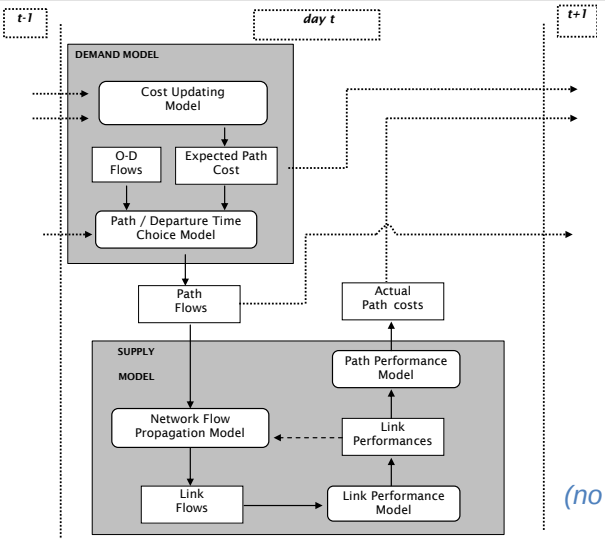
ASSIGNMENT MODELS
Congested network – Dynamic process models



(no information system)

ASSIGNMENT MODELS
Congested network – Dynamic process models

(no information – exponential smoothing example)

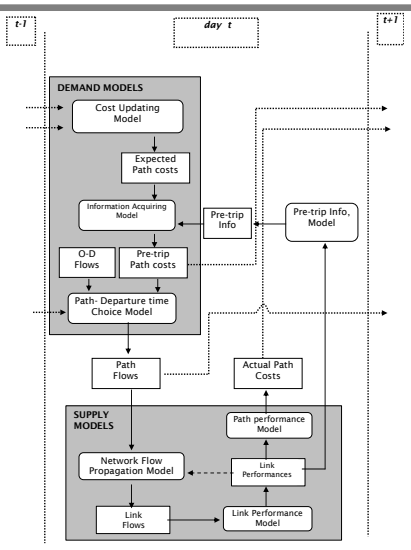
	WITHIN-DAY STATIC	WITHIN-DAY DYNAMIC
Supply	$\mathbf{c}_{act}^{t-1} = \mathbf{c}(\mathbf{f}^{t-1})$ $\mathbf{g}_{act}^{t-1} = \Delta^T \mathbf{c}_{act}^{t-1}$	$\mathbf{t}^{t-1} = \mathbf{t}(\mathbf{f}^{t-1})$ $\mathbf{TT}_{actj}^{t-1} = \Gamma(\mathbf{t}^{t-1})$
Demand	$\mathbf{g}_{pre}^{t} = \beta\, \mathbf{g}_{act}^{t-1} + (1-\beta)\, \mathbf{g}_{pre}^{t-1}$	$\mathbf{g}_{pre,j}^{t} = \beta[\mathbf{TT}_{act,j}^{t-1} + \mathbf{ELAD}_j(\mathbf{TT}_{act,j}^{t-1})] + (1-\beta)\, \mathbf{g}_{pre,j}^{t-1}$
	$\mathbf{h}^t = \alpha\, \boldsymbol{P}(\mathbf{g}_{pre}^t)\, d + (1-\alpha)\, \mathbf{h}^{t-1}$	$\mathbf{h}_j^t = \alpha\, \boldsymbol{P}_j(\mathbf{g}_{pre}^t)\, d + (1-\alpha)\, \mathbf{h}_j^{t-1}$
	$\mathbf{f}^t = \Delta\, \mathbf{h}^t$	$\mathbf{f}_j^t = \Sigma_{l \le j} \Delta_{l,j}(t_l, \ldots, t_j) \boldsymbol{P}_j(\mathbf{g}^t)\, \mathbf{d}$
	$\mathbf{c}_{act}^t = \mathbf{c}(\mathbf{f}^t)$	$\mathbf{t}_{act}^t = \mathbf{t}(\mathbf{f}^t)$
	$\mathbf{g}_{act}^t = \Delta^T \mathbf{c}_{act}^t$	$\mathbf{TT}_{act}^t = \Gamma(\mathbf{t}^t)$

ASSIGNMENT MODELS
Acronyms for dynamic process assignment models in the literature

DPA *Dynamic Process Assignment*

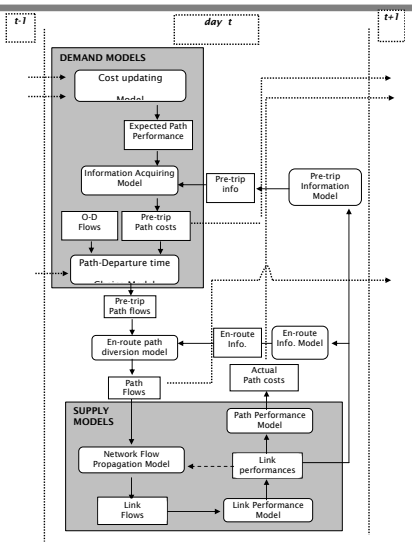
DPDA *Dynamic Process Dynamic Assignment*

ASSIGNMENT MODELS
Congested network – Dynamic process models



(pre-trip information system)

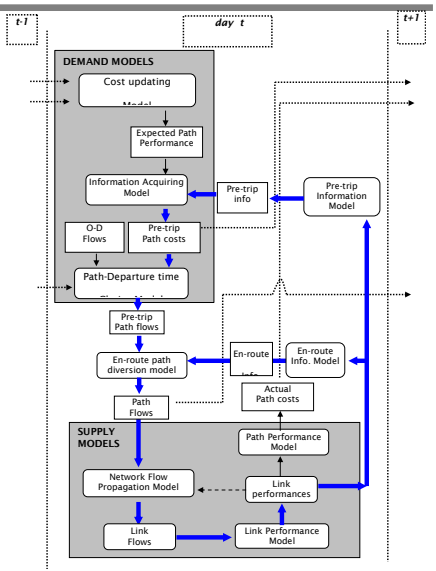
ASSIGNMENT MODELS
Congested network – Dynamic process models



(pre-trip /en-route information system)

ASSIGNMENT MODELS
Congested network – Dynamic process models

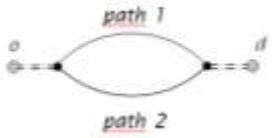
PREDICTIVE INFORMATION CONSISTENCY LOOPS



ASSIGNMENT MODELS
the concept of "over-reaction"

Providing predictive information not consistent with drivers' behavioral response (no consistency loop) may cause worsening of network performances

Example (i)



d_{od}=1000 veh/h

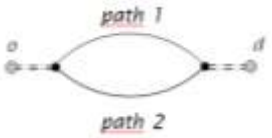
Usual network conditions

	free-flow travel time	free flow path choice probability	Equilibrium path choice probability	flow at equilibrium	Travel time at equilibrium
path 1	10	80%	70%	**700**	12
path 2	15	20%	30%	**300**	15
				1000	

ASSIGNMENT MODELS
the concept of "over-reaction"

Providing predictive information not consistent with drivers' behavioral response (no consistency loop) may cause worsening of network performances

Example (ii)



d_{od}=1000 veh/h

Market penetration (i.e. % of informed users) ***5%***

Accident occurring on path 1:
Travel time info on path 1: 12 ➔ 20 min

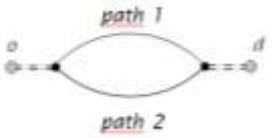
	Travel time at equilibrium	not informed drivers	informed drivers	total flow	travel time with accident
path 1	12	**665**		**665**	20
path 2	15	**285**	**50**	**335**	16
		950	**50**		

Travel time reduction for Informed Drivers: ***4 minutes***

ASSIGNMENT MODELS
the concept of "over-reaction"

Providing predictive information not consistent with drivers' behavioral response (no consistency loop) may cause worsening of network performances

Example (iii)



d_{od}=1000 veh/h

Market penetration (i.e. % of informed users) ***25%***

Accident occurring on path 1:
Travel time info on path 1: 12 ➔ 20 min

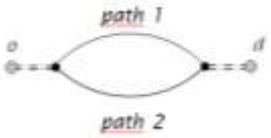
	Travel time at equilibrium	not informed	informed	total flow	travel time with accident
path 1	12	**525**		**525**	19
path 2	15	**225**	**250**	**475**	17
		750	**250**		

Travel time reduction: for Informed Drivers ***2 minutes***

ASSIGNMENT MODELS
the concept of "over-reaction"

Providing predictive information not consistent with drivers' behavioral response (no consistency loop) may cause worsening of network performances

Example (iv)



d_{od}=1000 veh/h

Market penetration (i.e. % of informed users) ***50%***

Accident occurring on path 1:
Travel time info on path 1: 12 ➔ 20 min

	Travel time at equilibrium	not informed	informed	total flow	travel time with accident
path 1	12	**350**		**350**	18
path 2	15	**150**	**500**	**650**	25
		500	**500**		

Travel time ***increase*** *for Informed Drivers:* ***7 minutes***

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Classification factors

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      Deterministic vs. Stochastic

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

The state in each period deterministically depends on previous states
(no information - within-day static example)

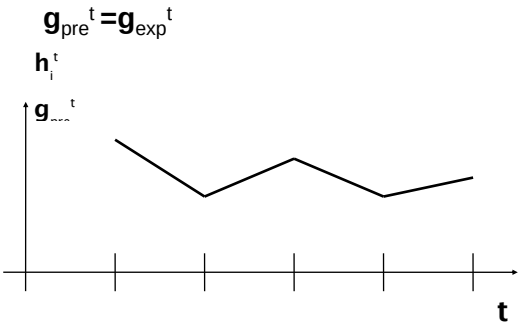
$\mathbf{g}_{pre}^{t} = \mathbf{g}_{exp}^{t}$

$\mathbf{g}_{pre}^{t} = g(\mathbf{g}_{act}^{t-1}, \mathbf{g}_{pre}^{t-1})$

$\mathbf{h}^{t} = \boldsymbol{P}(\mathbf{g}_{pre}^{t})\, \mathbf{h}^{t-1}$

$\mathbf{g}_{act}^{t} = \Delta^{T}\, \boldsymbol{c}(\Delta\, \mathbf{h}^{t})$

$\mathbf{f}^{t} = \Delta\, \mathbf{h}^{t}$



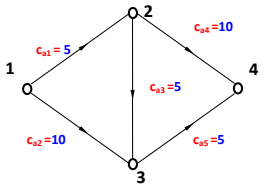
ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Example

Demand d(1,4)= 1500 veic/h

Free flow link cost



$\mathbf{g}_{pre}^{t} = 0{,}3\mathbf{g}_{act}^{t-1} + \mathbf{0{,}7}\mathbf{g}_{pre}^{t-1}$ (i.e. β=0,3)

$\mathbf{h}^{t} = \boldsymbol{P}(\mathbf{g}^{t})\,\mathbf{h}^{t-1}$ (i.e. α=1)

$$P(g_k^t) = \frac{\exp\left[-(1/60)\cdot g_k^t\right]}{\sum_{k'}\exp\left[-(1/60)\cdot g_{k'}^t\right]}$$

$\mathbf{g}_{act}^{t} = \Delta^{T}\,\boldsymbol{c}(\Delta\,\mathbf{h}^{t})$ $\quad c(f) = c_0+(0{,}005*f)^2$

$\mathbf{f}^{t} = \Delta\,\mathbf{h}^{t}$

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Example (I)

t=0	c_0	Path #	g^0_{act}	g^t_{pre}	P	h^t		f^t	$c(f^t)$
a1	5,0	1	-	15	33%	500	a1	1000	30,0
a2	10,0	2	-	15	33%	500	a2	500	16,3
a3	5,0	3	-	15	33%	500	a3	500	11,3
a4	10,0						a4	500	16,3
a5	5,0						a5	1000	30,0

t=1	c^{t-1}	Path #	g^{t-1}_{act}	g^t_{pre}	P	h^t		f^t	$c(f^t)$
a1	30,0	1	71,3	31,9	31%	459	a1	980	29,0
a2	16,3	2	46,3	24,4	35%	520	a2	520	16,8
a3	11,3	3	46,3	24,4	35%	520	a3	459	10,3
a4	16,3						a4	520	16,8
a5	30,0						a5	980	29,0

t=2	c^{t-1}	Path #	g^{t-1}_{act}	g^t_{pre}	P	h^t		f^t	$c(f^t)$
a1	29,0	1	68,3	42,8	29%	436	a1	968	28,4
a2	16,8	2	45,8	30,8	35%	532	a2	532	17,1
a3	10,3	3	45,8	30,8	35%	532	a3	436	9,7
a4	16,8						a4	532	17,1
a5	29,0						a5	968	28,4

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES
Example (I)

….

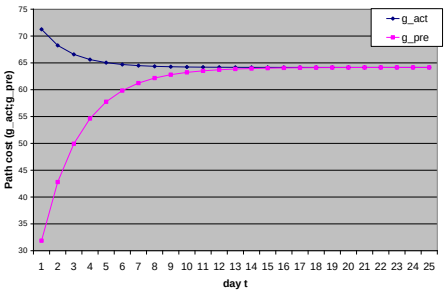
t=25	c^{t-1}	Path #	g^{t-1}_{act}	g^{t}_{pre}	P	h^{t}		f^{t}	$c(f^{t})$
a1	27,6	1	64,2	64,1	27%	400	a1	950	28
a2	17,6	2	45,1	45,1	37%	550	a2	550	18
a3	9,0	3	45,1	45,1	37%	550	a3	400	9
a4	17,6						a4	550	18
a5	27,6						a5	950	28

t=26	c^{t-1}	Path #	g^{t-1}_{act}	g^{t}_{pre}	P	h^{t}		f^{t}	$c(f^{t})$
a1	27,6	1	64,2	64,2	27%	400	a1	950	28
a2	17,6	2	45,1	45,1	37%	550	a2	550	18
a3	9,0	3	45,1	45,1	37%	550	a3	400	9
a4	17,6						a4	550	18
a5	27,6						a5	950	28

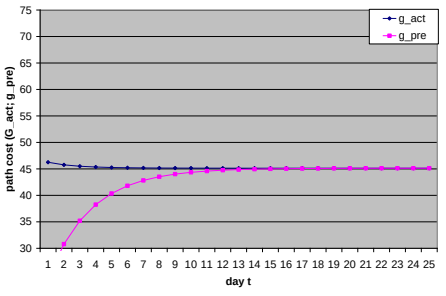
ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES
Example (I)
Actual vs Pre-trip Path cost day-to-day pattern

Path h1



Path h2, Path h3



ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Example (II): Actual vs Pre-trip Path cost day-to-day pattern

Example parameters (→ cfr. example I)

d(1,4)= 8000 veic/h ; β=0.77

t=0	c0	Path #	g_{act}	g_{pre}	P	h		f	c(f)
a1	5,0	1	15	15	33%	2667	a1	5333	716,1
a2	10,0	2	15	15	33%	2667	a2	2667	187,8
a3	5,0	3	15	15	33%	2667	a3	2667	182,8
a4	10,0						a4	2667	187,8
a5	5,0						a5	5333	716,1

t=1	$c(f^{k-1})$	Path #	g_{act}	g_{pre}	P	h		f_snl	$c(f^k)$
a1	716,1	1	1615,0	1247,0	0%	0	a1	4000	405,0
a2	187,8	2	903,9	699,4	50%	4000	a2	4000	410,0
a3	182,8	3	903,9	699,4	50%	4000	a3	0	5,0
a4	187,8						a4	4000	410,0
a5	716,1						a5	4000	405,0

t=2	$c(f^{k-1})$	Path #	g_{act}	g_{pre}	P	h		f	$c(f^k)$
a1	405,0	1	815,1	914,4	6%	462	a1	4231	452,5
a2	410,0	2	815,0	788,4	47%	3769	a2	3769	365,2
a3	5,0	3	815,0	788,4	47%	3769	a3	462	10,3
a4	410,0						a4	3769	365,2
a5	405,0						a5	4231	452,5

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Example (II)

….

t=25	$c(f^{k-1})$	Path #	g_{act}	g_{pre}	P	h		f	$c(f^k)$
a1	416,2	1	837,7	884,7	14%	1132	a1	4566	526
a2	399,0	2	815,2	818,1	43%	3434	a2	3434	305
a3	5,3	3	815,2	818,1	43%	3434	a3	1132	37
a4	399,0						a4	3434	305
a5	416,2						a5	4566	526

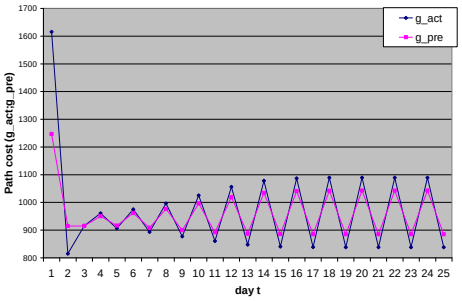
t=26	$c(f^{k-1})$	Path #	g_{act}	g_{pre}	P	h		f	$c(f^k)$
a1	526,2	1	1089,4	1042,3	1%	111	a1	4055	416
a2	304,8	2	831,0	828,0	49%	3945	a2	3945	399
a3	37,0	3	831,0	828,0	49%	3945	a3	111	5
a4	304,8						a4	3945	399
a5	526,2						a5	4055	416

ASSIGNMENT MODELS
Congested network – Dynamic process models

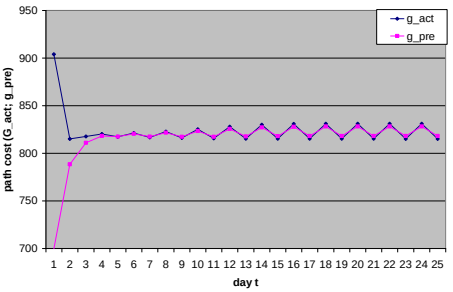
DETERMINISTIC PROCESSES

Example (II): Actual vs Pre-trip Path cost day-to-day pattern

Path h1



Path h2, Path h3



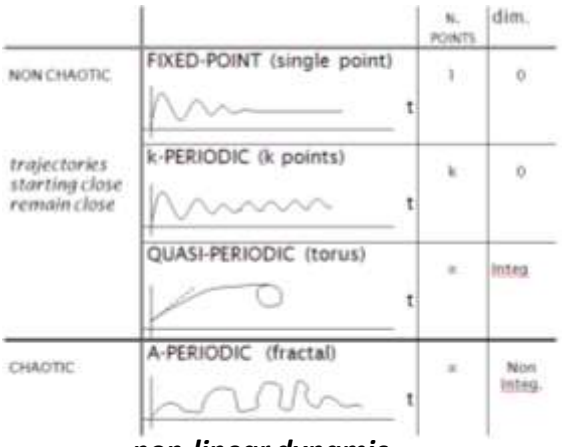
ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Attractors

A minimal subset of the state space:

- dimension less than the dimension of the state space
- the system may not evolve outside it, from a state inside it
- there is a proper superset (the basin) from which the system evolves toward the attractor



non-linear dynamic systems theory

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Stability condictions

The Stability of the attractor decreases when:

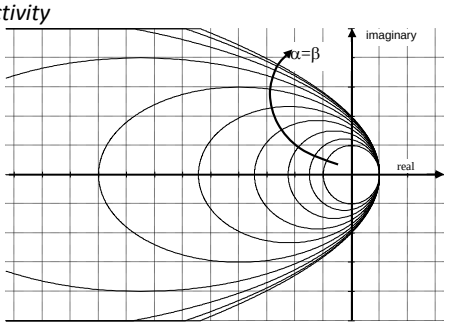
- ***demand flows*** *increase*
- ***link capacities*** *decrease*

} *Congestion levels increase*

- *the* ***variance of the random residuals*** *decreases* ➔ *more info*
- ***parameters*** α *and* β *increase* ➔ *more reactivity*

EXAMPLE

Stability regions (i.e. ellipses) of a fixed point state for $\alpha = \beta$.



α=β=1 ; 0.9 ; 0.8 ; 0.7 ; 0.6 ; 0.5 ; 0.4 ; 0.3 ; 0.2 ; 0.1

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES

Fixed-point attractors

(no information - within-day static example)

$$\mathbf{g}_{pre}^{t} = \mathbf{g}_{exp}^{t}$$

RECURSIVE EQUATIONS

$$\mathbf{g}_{pre}^{t} = g(\mathbf{g}_{act}^{t-1} = \Delta^{T}\boldsymbol{c}(\Delta\,\mathbf{h}^{t-1}),\ \mathbf{g}_{pre}^{t-1})$$
$$\mathbf{h}^{t} = \boldsymbol{P}(\mathbf{g}_{pre}^{t})\,\mathbf{h}^{t-1}$$

FIXED- POINT CONDITIONS

$$\mathbf{g}_{pre}^{t} = \mathbf{g}_{pre}^{t-1} = \mathbf{g}_{pre}^{*}$$
$$\mathbf{h}^{t} = \mathbf{h}^{t-1} = \mathbf{h}^{*}$$

⇩

FIXED POINT ATTRACTORS

$$\mathbf{g}_{pre}^{*} = g(\mathbf{g}_{act}^{*} = \Delta^{T}\,\boldsymbol{c}(\Delta\,\mathbf{h}^{*}),\ \mathbf{g}_{pre}^{*})$$
$$\mathbf{h}^{*} = \boldsymbol{P}(\mathbf{g}_{pre}^{*})\,\mathbf{h}^{*}$$

ASSIGNMENT MODELS
Congested network – Dynamic process models

DETERMINISTIC PROCESSES
Fixed-point attractors → EQUIVALENCE WITH EQUILIBRIUM IF:

• **HOMOGENEOUS COST UPDATING MODELS**
if the actual path cost at day *t-1 is different from what travellers expected:*

$$\mathbf{g}_{act}^{t-1} \neq \mathbf{g}_{pre}^{t-1}$$

such difference implies a different expected path cost at day t

$$\mathbf{g}_{pre}^{t} \neq \mathbf{g}_{pre}^{t-1}$$

• **EXPONENTIAL SMOOTHING CHOICE UPDATING MODELS**

$$\mathbf{g}_{act}^{*} = \Delta^{T} \boldsymbol{c}(\mathbf{d}\, \Delta\, \mathbf{h}^{*})$$
$$\mathbf{h}_{i}^{*} = \boldsymbol{P}(\mathbf{g}_{act}^{*})\, \mathbf{d}$$
$$\Downarrow$$
$$\mathbf{f}^{*} = \Delta\, \boldsymbol{P}(\Delta^{T}\boldsymbol{c}\, (\mathbf{f}^{*}))\, \mathbf{d}$$

ASSIGNMENT MODELS
Congested network – Dynamic process models

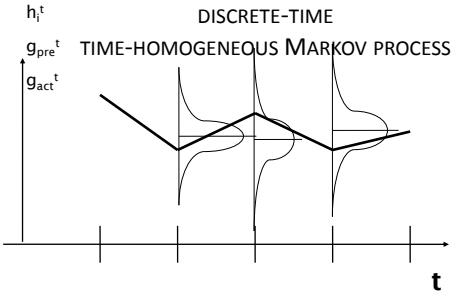
STOCHASTIC PROCESSES
State in each period is a random variable with distribution depending on previous states
(no information - within-day static example)

$\mathbf{g}_{pre}^{t} = \mathbf{g}_{exp}^{t}$

$\mathbf{g}_{act}^{t-1} \leftarrow \mathbf{G}_{act}^{t-1}$
with $E[\mathbf{G}_{act}^{t-1}] = \Delta^{T}\, \boldsymbol{c}(\Delta\, \mathbf{h}^{t-1})$

$\mathbf{g}_{pre}^{t} \leftarrow \mathbf{G}_{pre}^{t}$
with $E[\mathbf{g}_{pre}^{t}] = \boldsymbol{g}(\mathbf{g}_{act}^{t-1}, \mathbf{g}_{pre}^{t-1})$

$\mathbf{h}^{t} \leftarrow \mathbf{H}^{t}$
with $E[\mathbf{H}^{t}] = \boldsymbol{P}(\mathbf{g}_{pre}^{t})\, \mathbf{h}^{t-1}$



Properties of Stochastic Processes can be found in Cascetta (2009). *Transportation System Analysis: models and applications.* 2nd edition. Springer.

ASSIGNMENT MODELS
Congested network – Dynamic process models

EQUILIBRIUM VS. DYNAMIC PROCESSES
EQUILIBRIUM MODELS

ADVANTAGES
- no explicit modeling of users' cost and choice updating processes
- use of well founded models and algorithms (within-day static case)

DRAWBACKS
- uncertain relevance
- stability analysis not meaningful
- no simulation of transients and non recurrent conditions
- no system state statistical description

ASSIGNMENT MODELS
Congested network – Dynamic process models

EQUILIBRIUM VS. DYNAMIC PROCESSES
DYNAMIC MODELS

THEORETICAL ADVANTAGES
- identification of attractors
- stability analysis

APPLICATIVE ADVANTAGES
- simulation of transients and non recurrent conditions
- system state statistical description

DRAWBACKS
- require explicit modeling of users' cost and choice updating processes (memory, habit, etc.)
- computational

Dynamic control strategies reacting to perturbations in demand and/or supply can be effectively simulated only through dynamic process models